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Technical paper

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VARIATIONAL SHEAR CONTROL IN ESL–HSDT LAMINATES FOR AUTOMOTIVE STRUCTURES

ABSTRACT: An analytical formulation for the bending of laminated composite plates is developed based on the equivalent single-layer approach and higher-order shear deformation theory (ESL–HSDT). The key contribution lies in a variationally defined through-thickness shape function of exponential form, whose parameter is obtained by minimizing a functional governing the distribution of transverse shear deformation across the thickness. In this way, a direct link is established between the kinematic description and the distribution of shear energy, without increasing the kinematic order of the model. The solution is obtained in closed form using the Navier method for simply supported plates subjected to sinusoidal loading. The accuracy of the formulation is assessed through comparison with three-dimensional elasticity solutions and representative models from the literature. The results show that all considered formulations provide a comparable level of accuracy in predicting the global response, while the main differences arise in the transverse shear stresses. The proposed model yields the lowest overall deviation and provides a balanced and physically consistent distribution of shear energy through the thickness, confirming that the choice of the shape function represents a key mechanism for improving the accuracy of ESL–HSDT formulations.

KEYWORDS: Laminated plates, ESL–HSDT, Variational function, Shear energy, Navier solution

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VARIJACIONA KONTROLA SMICANJA U ESL–HSDT LAMINATIMA ZA AUTOMOBILSKE KONSTRUKCIJE

REZIME: U radu je razvijena analitička formulacija savijanja laminatnih kompozitnih ploča zasnovana na ekvivalentnom jednoslojnom pristupu i teoriji višeg reda (ESL–HSDT). Ključni doprinos predstavlja varijaciono definisana funkcija oblika kroz debljinu u eksponencijalnoj formi, čiji se parametar određuje minimizacijom funkcionala koji upravlja raspodelom transverzalne smičuće deformacije kroz debljinu. Na taj način uspostavlja se direktna veza između kinematičkog opisa i raspodele smičuće energije, bez povećanja kinematičkog reda modela. Rešenje je dobijeno u zatvorenoj formi primenom Navierovog postupka za slobodno oslonjene ploče pod sinusoidalnim opterećenjem. Tačnost formulacije procenjena je poređenjem sa trodimenzionalnim rešenjima teorije elastičnosti i reprezentativnim modelima iz literature. Rezultati pokazuju da sve razmatrane formulacije obezbeđuju uporediv nivo tačnosti u predviđanju globalnog odziva, dok se razlike između modela prvenstveno manifestuju u transverzalnim smičućim naponima. Predloženi model ostvaruje najmanje ukupno odstupanje i obezbeđuje uravnoteženu i fizički konzistentnu raspodelu smičuće energije kroz debljinu, čime se potvrđuje da izbor funkcije oblika predstavlja ključni mehanizam za unapređenje tačnosti ESL–HSDT formulacija.

KLJUČNE REČI: Laminatne ploče, ESL–HSDT, varijaciona funkcija, energija smicanja, Navierovo rešenje

INTRODUCTION

An analytical formulation for the bending of laminated composite plates is developed based on the equivalent single-layer approach and higher-order shear deformation theory (ESL–HSDT) (Reddy, J. N. 2003 and Karama, M. et al. 2003). The key contribution lies in a variationally defined through-thickness shape function of exponential form, whose parameter is determined by minimizing a functional governing the distribution of transverse shear deformation across the thickness. In this manner, a direct link is established between the kinematic description and the distribution of shear energy, without increasing the kinematic order of the model. In addition to these approaches, layerwise models have also been developed in the literature, enabling a more detailed representation of through-thickness quantities, albeit at the cost of increased computational complexity (Cetkovic, M. 2015). Furthermore, considerable attention has been devoted to the development of through-thickness shape functions, both through analytical approaches (Joshan, Y. S. et al. 2017; Radaković, A. et al. 2020 and Atallah, K. E. et al. 2025) and numerical methods (Ye, W. et al. (2026)). The solution is obtained in closed form using the Navier method for simply supported plates subjected to sinusoidal loading. The accuracy of the formulation is assessed through comparison with three-dimensional elasticity solutions and representative models from the literature (Reddy, J. N. 2003; Karama, M. et al. 2003 and Radaković, A. et al. 2020). The results show that all considered formulations provide a comparable level of accuracy in predicting the global response, while the main differences arise in the transverse shear stresses. The proposed model yields the lowest overall deviation and provides a balanced and physically consistent distribution of shear energy through the thickness, confirming that the choice of the shape function represents a key mechanism for improving the accuracy of ESL–HSDT formulations. In this context, the present study introduces a variationally defined through-thickness shape function that enables a consistent control of shear deformation distribution without increasing the kinematic complexity of ESL–HSDT formulations.

2. THEORETICAL FORMULATION

An equivalent single-layer higher-order model with seven generalized unknown functions, u_0, v_0, w_b, w_s, θ_x, θ_y, and ϕ, is considered. The kinematic field is described in terms of the mid-plane displacements $u_0(x,y)$, $v_0(x,y)$, the bending and shear components of the transverse displacement $w_b(x,y)$, $w_s(x,y)$, the rotations $\theta_x(x,y)$, $\theta_y(x,y)$, and an additional scalar function $\phi(x,y)$accounting for transverse normal deformation.

The displacement field is defined as:

$$u(x,y,z) = u_0 - z\left(w_{b,x} + w_{s,x}\right) + F(z)\left(w_{s,x} + \theta_x\right),\ v(x,y,z) = v_0 - z\left(w_{b,y} + w_{s,y}\right) + F(z)\left(w_{s,y} + \theta_y\right),$$
$$w(x,y,z) = w_b + w_s + \phi F'(z) \tag{1}$$

This representation enables an independent description of bending and shear contributions to the transverse displacement, while the additional term involving ϕintroduces transverse normal deformation without increasing the kinematic order.

The distribution of transverse deformation is governed by the through-thickness shape function $F(z)$, whose first and second derivatives directly enter the expressions for shear and transverse normal strains. In contrast to conventional approaches where this function is prescribed a priori, it is here defined in a variational form, allowing for a controlled distribution of shear deformation across the thickness.

Introducing the dimensionless coordinate $\zeta = z/h$, the following auxiliary function is adopted:

$$E = e^{-a\zeta^2} \tag{2}$$

and the shape function is expressed as:

$$F(z) = h\left[C\ erf(\sqrt{a}\zeta) + \frac{2}{a}\zeta E \right]. \tag{3}$$

Its derivatives are given by:

$$F'(z) = \left(1 - 4\zeta^2\right)E, \quad F''(z) = \frac{1}{h}E\left[-8\zeta - 2a\zeta\left(1 - 4\zeta^2\right)\right]. \tag{4}$$

The parameter α is determined variationally, linking the assumed kinematics to the through-thickness shear energy distribution and eliminating the need for predefined shear profiles while preserving the regularity of the displacement field. The strain field is derived from the assumed displacement field using the standard relations of linear elasticity. In accordance with the adopted kinematics, the resulting expressions naturally decompose into membrane, bending, and shear contributions, while the through-thickness distribution is governed by the shape function $F(z)$and its derivatives.

The in-plane normal strains are given by

$$\varepsilon_{xx} = u_{0,x} - z\left(w_{b,xx} + w_{s,xx}\right) + F(z)\left(w_{s,xx} + \theta_{x,x}\right),\ \varepsilon_{yy} = v_{0,y} - z\left(w_{b,yy} + w_{s,yy}\right) + F(z)\left(w_{s,yy} + \theta_{y,y}\right), \tag{5}$$

where the first term represents the membrane contribution, the linear term in $z$corresponds to bending, and the term involving $F(z)$accounts for the shear-related correction.
The in-plane shear strain is expressed as

$$\gamma_{xy} = u_{0,y} + v_{0,x} - 2z\left(w_{b,xy} + w_{s,xy}\right) + F(z)\left(2w_{s,xy} + \theta_{x,y} + \theta_{y,x}\right). \tag{6}$$

The transverse shear strains take the form

$$\gamma_{xz} = F'(z)\left(w_{s,x} + \theta_x\right), \quad \gamma_{yz} = F'(z)\left(w_{s,y} + \theta_y\right) \tag{7}$$

By introducing the additional scalar function $\phi(x, y)$, the transverse normal strain can be expressed as

$$\varepsilon_{zz} = \phi F''(z) \tag{8}$$

which extends the formulation to a quasi-three-dimensional level and enables a consistent evaluation of the stress state within the framework of linear elasticity.

In each lamina, the stress state is defined by an orthotropic constitutive relation in the local material coordinate system

$$\boldsymbol{\sigma}^{(k)} = \mathbf{Q}^{(k)}\boldsymbol{\varepsilon}^{(k)}, \tag{9}$$

where $\mathbf{Q}^{(k)}$denotes the stiffness matrix of the k-th layer.

The overall stress state is obtained by evaluating this relation at each point through the laminate thickness, where discontinuities in $\bar{\mathbf{Q}}^{(k)}$naturally arise between adjacent layers due to differences in fiber orientation and material properties.

Starting from the assumed displacement field and the corresponding strain components, the equilibrium equations in variational form are derived using the principle of virtual work. For a laminated plate, the following condition holds:

$$\delta\Pi = \delta U - \delta W = 0, \tag{10}$$

where δUdenotes the virtual work of internal forces and $\delta W$the virtual work of external loading.
The virtual work of internal forces is expressed as the sum of contributions from all layers through the thickness:

$$\delta U = \sum_{k=1}^{N} \int_A \int_{z_k}^{z_{k+1}} \delta \boldsymbol{\varepsilon}^T \bar{\mathbf{Q}}^{(k)} \boldsymbol{\varepsilon} dz dA \tag{11}$$

where $\boldsymbol{\varepsilon}$and $\delta\boldsymbol{\varepsilon}$are the strain and virtual strain vectors, respectively, and $\bar{\mathbf{Q}}^{(k)}$is the transformed stiffness matrix of the k-th layer in the global coordinate system.

For a sinusoidal transverse load $q(x, y)$, the virtual work of external forces is given by

$$\delta W = \int_A q(x,y) \delta w dA \tag{12}$$

Substituting the kinematic relations into δU and grouping terms with respect to the generalized unknowns {u_0, v_0, w_b, w_s, θ_x, θ_y, ϕ} yields a variationally consistent system with seven unknown functions. For simply supported plates, Navier's method is applied, expanding the unknowns in double Fourier series satisfying the boundary conditions. Substitution into the variational form and use of trigonometric orthogonality reduce the problem to a system of algebraic equations for each mode pair (m, n).

$$\mathbf{K}_{mn} \mathbf{d}_{mn} = \mathbf{f}_{mn}, \tag{13}$$

where $\mathbf{d}_{mn} = [U_{mn} \quad V_{mn} \quad W_{mn}^b \quad W_{mn}^s \quad \Theta_{mn}^x \quad \Theta_{mn}^y \quad \Phi_{mn}]^T$.

The matrix $\mathbf{K}_{mn}$contains contributions from membrane, bending, shear, and transverse normal deformation, with its coefficients obtained through integration across the laminate thickness. The shape function $F(z)$, together with its derivatives $F'(z)$and $F''(z)$, enters the system coefficients directly, thereby establishing a link between the adopted kinematics and the energetic response of the model.

3. NUMERICAL RESULTS AND DISCUSSION

To validate the proposed formulation, a simply supported laminated plate (SSSS) subjected to a sinusoidal transverse load is considered:

$$q(x,y) = q_0 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right), \tag{14}$$

which enables the use of a Navier-type solution.

Consistent with the adopted formulation, the generalized unknowns are expanded in double Fourier series compatible with the boundary conditions and loading pattern:

$$\begin{aligned}
&u_0(x,y) = \sum_{m,n} U_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right), \; v_0(x,y) = \sum_{m,n} V_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right), \\
&w_b(x,y) = \sum_{m,n} W_{mn}^b \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right), \; w_s(x,y) = \sum_{m,n} W_{mn}^s \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right), \\
&\theta_x(x,y) = \sum_{m,n} \Theta_{mn}^x \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right), \; \theta_y(x,y) = \sum_{m,n} \Theta_{mn}^y \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right), \\
&\phi(x,y) = \sum_{m,n} \Phi_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right).
\end{aligned} \tag{15}$$

A symmetric four-layer laminate with stacking sequence $[0/90/90/0]$is considered. The following dimensionless material properties are adopted:

$$E_1 / E_2 = 25, \quad G_{12} / E_2 = 0.5, \quad G_{13} / E_2 = 0.5, \quad G_{23} / E_2 = 0.2, \nu_{12} = \nu_{13} = \nu_{23} = 0.25.$$

Material anisotropy is accounted for through the transformed stiffness matrices $\bar{\mathbf{Q}}^{(k)}$.

For comparison with reference solutions, the results are presented in dimensionless form:

$$\bar{w} = w\left(\frac{a}{2},\frac{b}{2},0\right)\frac{100E_2h^3}{q_0a^4},\ \bar{\sigma}_{xx} = \sigma_{xx}\left(\frac{a}{2},\frac{b}{2},\frac{h}{2}\right)\frac{h^2}{q_0a^2},\ \bar{\sigma}_{yy} = \sigma_{yy}\left(\frac{a}{2},\frac{b}{2},\frac{h}{4}\right)\frac{h^2}{q_0a^2}$$
$$\bar{\tau}_{xz} = \tau_{xz}\left(0,\frac{b}{2},0\right)\frac{h}{q_0a},\ \bar{\tau}_{yz} = \tau_{yz}\left(\frac{a}{2},0,0\right)\frac{h}{q_0a}. \tag{16}$$

This normalization allows for a direct comparison with three-dimensional elasticity solutions and representative results available in the literature.

Table 1. Comparison of normalized displacements and stress components with 3D elasticity solutions, with percentage deviations for different models

Model	w̄	Diff (%)	σ̄xx	Diff (%)	σ̄yy	Diff (%)	τ̄xz	Diff (%)	τ̄yz	Diff (%)	σ̄xy	Diff (%)	Avg Diff (%)
3D	1.954	–	0.720	–	0.663	–	0.219	–	0.291	–	0.047	–	–
Karama	1.920	-1.72	0.708	-1.65	0.639	-3.68	0.253	+15.57	0.227	-22.16	0.046	-2.13	7.8
Reddy	1.895	-3.04	0.673	-6.60	0.634	-4.34	0.239	+9.09	0.206	-29.07	0.044	-6.17	9.7
Present	1.925	-1.47	0.736	+2.22	0.639	-3.63	0.264	+20.46	0.244	-16.25	0.047	+0.64	7.4
Radaković	1.889	-3.31	0.667	-7.32	0.633	-4.48	0.237	+8.08	0.204	-30.07	0.044	-6.81	10.0

Table 1 presents a comparison between the obtained results and reference three-dimensional elasticity solutions. The relative deviations are expressed in percentage form as

$$\text{Diff}(\%) = \frac{X_{\text{model}} - X_{3D}}{X_{3D}} \times 100. \tag{17}$$

Results indicate that all considered formulations achieve close agreement with the reference solution in terms of global response, particularly for deflections and normal stresses σ_{xx} and σ_{yy}, with deviations remaining limited in magnitude. In contrast, more pronounced differences are observed in the transverse shear stresses, highlighting the dominant influence of the through-thickness shape function on the local stress state.

From an overall accuracy standpoint, the present model exhibits the most favourable balance across all response components, with a stable behaviour and no pronounced local deviations. The results suggest that the exponentially defined shape function enables a more flexible redistribution of shear energy, which translates into a more consistent agreement with the reference solution. The Karama model provides comparable predictions for normal stresses, but shows more pronounced deviations in the shear components, indicating a limited capability to accurately capture the through-thickness distribution. The Reddy and Radaković formulations exhibit a stable yet systematically conservative response in shear, with a tendency to underestimate peak values.

The largest discrepancies are observed in the τ_{yz} component, confirming that the representation of transverse shear deformation remains the primary challenge for equivalent single-layer models. This clearly demonstrates that the choice of shape function primarily governs the distribution of shear energy, while its influence on global response quantities is significantly less pronounced. The through-thickness distributions of normalized stress components for different shape functions are shown in Fig. 1. The normal stresses $\sigma_{xx}$and $\sigma_{yy}$are largely insensitive to the choice of shape function, indicating their dominant dependence on the global bending response. A similar trend is observed for σ_{xy}, confirming a consistent representation of membrane–bending coupling across all formulations. In contrast, the transverse shear stresses $\tau_{xz}$and τ_{yz}exhibit a pronounced dependence on the form of $F(z)$, with the largest differences occurring in the vicinity of the mid-plane. The present formulation produces a more pronounced yet smooth distribution of shear stresses in the central region of the laminate, without spurious oscillations or abrupt gradient changes. Compared to the other models, this behaviour indicates a more stable and physically consistent description of through-thickness response. By contrast, formulations based on more rigid shape functions (Reddy, Radaković) yield more diffuse distributions with reduced peak values, while the Karama model exhibits a faster decay through the thickness, limiting its ability to capture local gradients. These results highlight that the choice of shape function directly governs the quality of shear stress prediction, with the present model achieving a balanced representation that combines local adaptability with global stability.

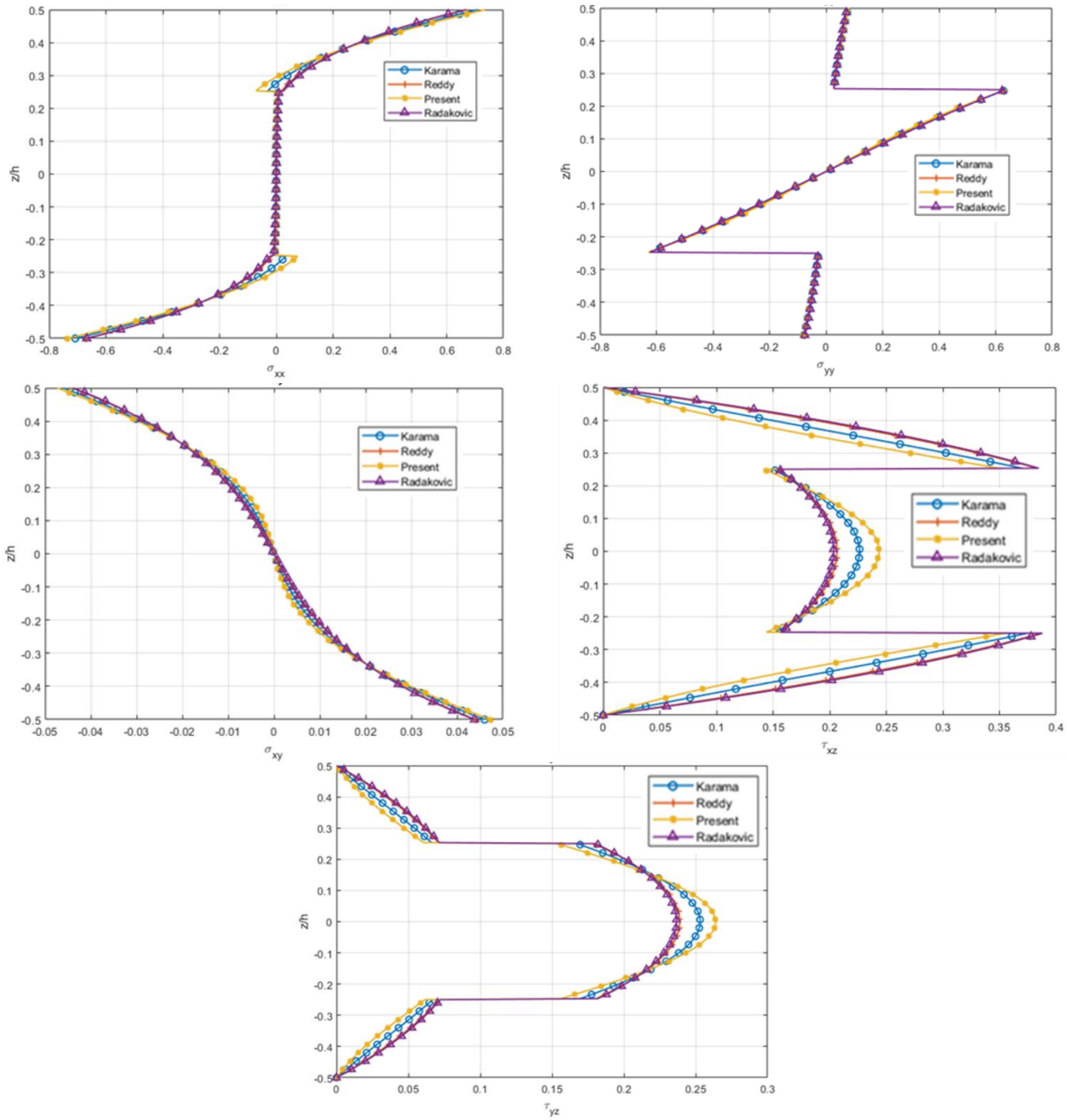


Figure 1 Through-thickness variation of normalized stress components (σ_{xx}, σ_{yy}, σ_{xy}, τ_{xz}, τ_{yz}) for different ESL–HSDT models

CONCLUSIONS

The proposed formulation enables a consistent description of through-thickness behaviour of laminated plates within the ESL–HSDT framework, without increasing kinematic complexity. The results show that differences between formulations do not stem from the global response, but from the manner in which transverse shear deformation is represented.

Numerical results confirm that all considered models provide comparable accuracy in predicting global quantities, while their differences are primarily reflected in the distribution of transverse shear stresses. The through-thickness analysis indicates that the choice of shape function directly governs the character of the local response, with different formulations effectively corresponding to different regimes of shear deformation distribution.

The present model exhibits a stable and consistent response across the entire thickness, without spurious oscillations and with a smooth stress field. This behaviour suggests that improving the description of shear effects does not require additional kinematic variables, but rather a more precise control of the through-thickness deformation field.

From a broader perspective, the findings indicate that further development of ESL–HSDT models should focus on the formulation of the shape function rather than on increasing the order of the theory. This provides a viable path toward more efficient modeling of laminated structures in applications where through-thickness effects are critical.

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